Resit Exam: Sets and Numbers 2025

Numbers part:

0. Let $a_n = (3^{2n} - 8n - 1)/64$. Show that $a_n \in \mathbb{Z}$ for all integers n.

1. Prove that, if a and b are relatively prime, then
$$\gcd(2a + b, a + 2b) = 1 \text{ or } 3.$$

2. *Prove or disprove.* If $a^3 \mid c^2$ then $a \mid c$.

3. a) For which values of k does the equation $315x + 56y = k$ have at least one integer solution?
   b) Solve the linear congruence $315x \equiv 24 \pmod{56}$, or explain why no solutions exist.

4. Prove that $437 = 19 \cdot 23$ divides $3^{396} - 1$.

5. Show that if a and n are not relatively prime, then a has no multiplicative inverse modulo n.

6. *Prove or disprove.* The matrix equation over $\mathbb{Z}/35\mathbb{Z}$
$$\begin{pmatrix} \bar{3} & \bar{2} \\ \bar{7} & \bar{6} \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} \bar{1} \\ \bar{4} \end{pmatrix}$$
has a unique solution.

7. Show that
$$1^p + 2^p + 3^p + \ldots + (p-1)^p \equiv 0 \pmod{p}$$
when p is an odd prime.

8. Find four positive integers n such that $\varphi(n) = 24$ where φ denotes the Euler totient function.